Branching interfaces with infinitely strong couplings

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Abstract. A hierarchical froth model of the interface of a random q-state Potts ferromagnet in 2D is studied by recursive methods. A fraction p of the nearest-neighbour bonds is made inaccessible to domain walls by infinitely strong ferromagnetic couplings. Energetic and geometric scaling properties of the interface are controlled by zero-temperature fixed distributions. For $p < p_c$, the directed percolation threshold, the interface behaves as for $p = 0$, and scaling supports random Ising ($q = 2$) critical behaviour for all q's. At $p = p_c$ three regimes are obtained for different rates of ferro versus antiferromagnetic couplings. With rates above a threshold value the interface is linear (fractal dimension $d_f = 1$) and its energy fluctuations, ΔE scale with length as $\Delta E \propto L^\omega$, with $\omega \simeq 0.48$. When the threshold is reached the interface branches at all scales and is fractal ($d_f \simeq 1.046$) with $\omega_c \simeq 0.51$. Thus, at p_c, dilution modifies both low-temperature interfacial properties and critical scaling. Below threshold the interface becomes a probe of the backbone geometry ($d_f \simeq \bar{d} \simeq 1.305$; $\bar{d}$ = backbone fractal dimension), which even controls energy fluctuations ($\omega \simeq d_f \simeq \bar{d}$). Numerical determinations of directed percolation exponents on diamond hierarchical lattice are also presented.

1. Introduction

The effects of quenched disorder due to random impurities on phase transitions are often non-trivial and have long represented an active field of research. According to a simple Harris criterion [1], disorder is relevant to continuous phase transitions with positive specific heat exponent. The presence of impurities can also change first-order transitions into second order, and typically has rounding effects upon them [2, 3]. Numerical and experimental evidence exists [4, 5] that in some random systems, like Potts ferromagnets, the critical behaviour falls into the random-bond Ising model (RBIM) universality class, irrespective of the different symmetry they possess. Recently, a justification of this possible universality has been proposed, based on the analysis of the behaviour of critical interfaces in the presence of strong disorder [6].

The interfacial free energy is indeed crucial to second-order phase transitions since it vanishes in a singular way at the critical point. RBIM interfaces have been successfully investigated within the solid-on-solid (SOS) model, which ignores overhangs and islands near the interface [7]. This model is equivalent to the directed polymer in random medium (DPRM) [8], which reduces to an optimization problem of path energies, at $T = 0$.

The q-state Potts model generalizes the Ising one with its S_q symmetry. When the model undergoes continuous phase transitions, Potts interfaces are complicated branched objects which become fractal at criticality, in the pure case. The so-called 'froth model' has been introduced as a simplified description of those interfaces [6]. This model represents a non-trivial generalization of the DPRM and provides a possible mechanism for the explanation of

the puzzling universality in the strongly disordered Potts model at criticality. The realization of the froth model on a diamond hierarchical lattice (DHL) in the presence of coupling constant randomness is the only context where the hard computational difficulties connected with its study could be successfully faced so far. Indeed, on DHL the model is amenable to an accurate position-space renormalization-group analysis yielding deep insight into interfacial scaling properties [6].

A variant to the DPRM has been recently obtained through the introduction of a geometrical disorder represented by bond dilution [9], which shows itself to be an implicit source of randomness for path energy distributions. Dilution, conceived as the presence of a fraction of bonds inaccessible by the polymer, modifies or leaves unaffected DPRM critical behaviour depending on whether the concentration of present (accessible) bonds is at or above the directed percolation threshold, respectively. The structure of the underlying directed bond percolation cluster governs the scaling behaviour of the model right at the critical threshold [9].

Interest in investigating the effects of a similar dilution on the froth interface model of [6] is suggested by different considerations. As the DPRM can describe fracture [10], the froth interface can also be seen as realizing a more complicated pattern of fracture, with the possibility of branching and loop formation in the cracks. Dilution can thus represent in both cases the effects of hard inclusions through which cracks cannot propagate, in an inhomogeneous material. In some systems like ^{3}He–^{4}He mixtures in the pores of aerogels, dense coatings of ^{4}He are formed on the surface of the aerogel structure [11]. The higher density considerably enhances the interaction between ^{4}He atoms. This effect adds to a more general introduction of disorder in the interactions. One expects that enhanced exchange interactions around the aerogel prevent penetration of the interface associated with the superfluid transition in the aerogel†. The space regions occupied by the aerogel and its coating can again be seen as inclusions with very hard coupling for the interface.

These examples suggested we consider a diluted generalization to the froth interfacial model of [6], by allowing lattice bonds to be absent, i.e. inaccessible to the interface, with a finite probability. As in [9], here we choose a percolative dilution geometry, which is of course rather different from that of the aerogels case.

Dilution, especially at percolation threshold, is an obvious candidate to possibly introduce new universality classes of interfacial scaling behaviour. This is most important for branching interfaces in a random environment, because their study is still at a very preliminary stage [6].

A further motivation for diluting the froth interface model comes from the legitimate hope that some of its scaling properties at the percolation threshold could be more or less directly related to the geometry of the critical percolation cluster.

This paper is organized as follows. In section 2 we introduce the froth model of the Potts interface and set up an iterative renormalization-group (RG) analysis for its study on hierarchical lattice. We discuss, in particular, the specific features implied by the presence of dilution. In section 3 we present our results for several regimes of dilution. Section 4 is devoted to a summary and to concluding remarks.

2. Diluted model of the Potts interface

The interface between two distinct ordered phases of a 2D random-bond q-state Potts ferromagnet at very low temperature is a slightly fluctuating line. Raising the temperature

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† In a Bulme–Emery–Griffiths description (see [11]) this interface is like an Ising $+/-$ domain wall.
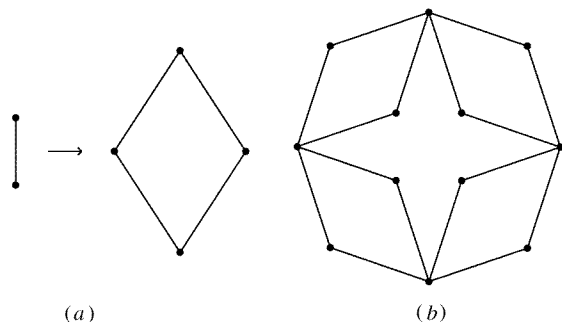
Figure 1. (*a*) The construction rule of the diamond hierarchical lattice. At every stage each bond is replaced by a four-bond cell. (*b*) The DHL at its second stage.

to approach its critical value, bubbles of any of the other $q - 2$ states may form at the interface; neglecting isolated islands and overhangs, as in SOS models of Ising surfaces, we are left to consider an aggregate of bubbles, the froth, each bounded by two SOS surfaces. This froth represents the original interface. In a random-bond Potts problem the interface crosses (breaks) bonds with random couplings, thus getting a random energy contribution at each step. In addition to random couplings, here we suppose there exist, with probability p, steps inaccessible to the interface (dilution).

Even starting from the above simplified picture, the corresponding interfacial model is too hard to analyse. By confining the interface to the bonds of the DHL shown in figure 1, we obtain a simplified version of the froth model, where a position space RG strategy can be worked out. This model was already studied in [6] in the undiluted case ($p = 0$), with only random exchange effects.

The interface partition function, Z_n, on the diamond hierarchical lattice at its nth construction stage (longitudinal length $L = 2^n$) is calculated iteratively. In the pure case (no randomness and no dilution),

$$Z_{n+1} = 2Z_n^2 + (q - 2)Z_n^4 \qquad (2.1)$$

where $Z_0 = e^{-\beta 2J}$ is the Boltzmann weight of a broken bond, and $\beta = 1/k_B T$. The entropic factor $q - 2$ accounts for the number of different coexisting phases which can occupy a bubble. An analysis of the RG flow described by equation (2.1) shows that there is a finite unstable fixed point Z_c, besides $Z = 0$ and $Z = \infty$. The quantity $f = \lim_{n\to\infty} \ln(Z_n)/L$ should be < 0 and coincide with the interfacial line tension for $T < T_c$. So, the region $Z < Z_c$ corresponds to the low-temperature regime, because there $Z = \exp(Lf)$, where f ($f < 0$) is the line tension. For $Z > Z_c$ the line tension description does not apply, since upon iterating equation (2.1) Z approaches infinity as $\exp(L^2 f_b)$, where $f_b > 0$ is a sort of dense froth free energy which describes the system in the high-T phase. The RG analysis provides the critical Boltzmann weight, Z_c, and the interface critical exponent $\mu(q)$, characterizing the vanishing of the interfacial free energy, $f \sim |T - T_c|^\mu$, for any q†. Indeed, if we put $t_n = Z_n - Z_c$, we get by construction $f(t_0) = 2^{-1} f(t_1) = 2^{-1} f(2^{y_T} t_0)$, which leads to the expected scaling of f for $t \to 0$, with $\mu = \ln(\partial t_1/\partial t_0)/\ln 2 = y_T$. By keeping the lengths of the system finite and equal to L and $L' = L/2$, we also obtain

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† As already stressed in [6], this approach is not able to show the expected change to first-order phase transition at $q = 4$.

the finite-size scaling (FSS) version of the previous law: $f(t_0, L) = 2^{-1}f(2^{y_T}t_0, L')$. Of course, the thermal exponent y_T also has the meaning of the fractal dimension of the critical interface because the average number of broken bonds is obtained by differentiating $\ln Z_n$ with respect to t_0.

In the presence of random couplings and bond dilution, Z_n is a random variable whose distribution function, $P_n(Z)$, has to be iterated starting from the initial form

$$P_0(Z) = p\delta(Z) + (1-p)\mathcal{P}_0(Z). \tag{2.2}$$

Namely each accessible bond is given a weight, Z, according to $\mathcal{P}_0(Z)$. p is the probability that a given bond is not accessible to the interface. For a given realization of disorder (2.2) in the whole structure, the partition function evolves from one stage to the next as

$$Z_{n+1} = Z_n(1)Z_n(2) + Z_n(3)Z_n(4) + (q-2)\prod_{i=1}^{4}Z_n(i) \tag{2.3}$$

where the indices 1 to 4 refer to the four n-order elements constituting an $(n+1)$-order diamond. $P_n(Z)$ iterates according to the rules of the nonlinear composition of independent random variables defined in (2.3). Fixed points of the mapping (2.1) are replaced by fixed distributions.

It is easy to see that the component of P_n concerning the probability for a bond to be infinitely hard decouples from the other quantities and iterates separately [9] as

$$p_{n+1} = \left[1 - (1-p_n)^2\right]^2 \tag{2.4}$$

where p_n is the probability for inaccessible macrobonds to occur through the n-level lattice. So equation (2.4) is nothing but the RG transformation for percolation on DHL, whose critical fixed point is at $p_c = (3 - \sqrt{5})/2$ [9]. As we will also discuss at the end of the next section, this is a reasonably good qualitative model of directed percolation in 2D.

The second component, $\mathcal{P}_0(Z)$, of $P_0(Z)$ is related to energy randomness, and couples to bond dilution under RG flow. Hence even in non-random energy cases, $\mathcal{P}_0(Z) = \delta(Z - Z_0)$, Z_1 no longer possesses a binary distribution with values 0 and Z, even after the first iteration. Thus $P_n(Z)$ always evolves towards a multivalued distribution function, like in the random energy model.

In our model we verified that both dilution and disorder are relevant perturbations of the pure system fixed point of (2.1). Furthermore, similarly to what happens in the undiluted case [6], the impure system behaviours turn out to be always controlled by $T = 0$ fixed-point distribution functions. This is similar to what happens in analogous problems [8, 12], where randomness is relevant and energy optimization criteria prevail with respect to entropic considerations. Dilution, here, is a further source of disorder. So it is no surprise that the dilute random system is governed by zero-temperature strong-disorder fixed-point distributions for all dilution regimes. We examine properties of these distributions in detail below by considering from the start $T = 0$ RG recursions.

We are then concerned with the iteration of the interfacial energy distribution starting from one which introduces absent (i.e. inaccessible) bonds with probability p and present (i.e. accessible) ones with probability $1 - p$. The energies e of the latter at the $n = 0$ level are chosen according to a probability density, $\mathcal{P}(e) = u\delta(e-1) + (1-u)\delta(e+1)$. In the pure system we had $e = 2J$ for all bonds. By decreasing u we expect to reach criticality conditions similar to those found by rising T at non-zero temperature [6]. On
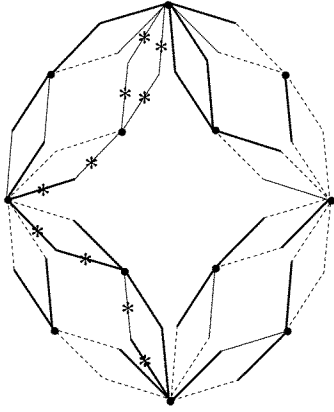
Figure 2. Diluted DHL at its third growth stage. A particular configuration of the froth obtained at $p = p_c$ and $u = u_c$ is illustrated. Broken lines represent inaccessible bonds, full lines mark accessible bonds with either energy $e = +1$ (thick lines), or $e = -1$ (thin lines). Notice the presence of a critical backbone connecting upper and lower ends of the DHL. At $T = 0$ the froth steps on the minimal energy connected configuration, as indicated by the asterisks.

a given sample, the interface at level $n + 1$ can cross either of the two lattice branches made by two linked n-macrobonds, or form a bubble through both. The choice among these three possibilities is only dictated by the criterion of energy minimization, provided the n-macrobonds are globally accessible. Figure 2 gives an illustration of such a DHL interface model at the critical dilution, $p = p_c$. Monte Carlo iteration of the energy distribution density is relatively simple and fast†. Starting with a large sample ($\sim 2 \times 10^5$) of bond energies we could iterate it up to $n \sim 20$ without propagating too large statistical uncertainties due to finite samplings of the distributions.

3. Numerical and exact results on RG flows

RG flows confirm the existence of a critical percolation threshold for absent bonds at $p = p_c$. This is indeed the threshold above which the interface can no longer cross the system, unless at the cost of an infinite positive energy, even for finite system sizes. The numerical value $p_c = 0.3820 \pm 0.0008$ compares quite well with the theoretical one. As in [9] we find the influence of dilution on the interfacial critical system to be qualitatively different depending on whether $p < p_c$, or $p = p_c$.

In the $p < p_c$ regime forbidden regions do not percolate and only isolated finite clusters of inaccessible bonds exist. Clusters are fractal on scales shorter than their linear size. On large length scales they are homogeneously distributed and should not exert any particular influence on the interface, except as an additional source of disorder. We observe in the interface two distinct phases separated by a critical value, u_c, of the disorder parameter u ($u/(1-u)$ gives the ratio of ferro- versus antiferromagnetic accessible bonds). u_c of course depends on p and on the fact that we were setting to unity the starting absolute e value.

When $u > u_c$ the interface is linear, its energy distribution has a mean value, $\langle E(L) \rangle$, which shifts as $\langle E(L) \rangle \simeq \langle E_0 \rangle L \gg 0$ ($L = 2^n$ is the length of the DHL at level n) and a width, $\Delta E(L)$, growing as L^ω, with $\omega \approx 0.30$. The interface is thus a directed path on a DHL with the energy distribution discussed in [13]. For $u < u_c$ we find a dense foam of bubbles, whose energy density function has mean value $\langle E(L) \rangle \propto -L^2$ and width $\Delta E(L) \propto L$. The central limit theorem applies to this distribution function which should be asymptotically Gaussian. At $u = u_c$ we have $\langle E(L) \rangle \propto L^{y_c}$ and $\Delta E(L) \propto L^{\omega_c}$, with

† Starting form an ensemble consistent with $\mathcal{P}(e)$, at the $(n + 1)$-th iteration groups of four n-macrobonds are sampled and the least path-energy is chosen to be the actual value of that $(n + 1)$-macrobond.

$y_c \simeq \omega_c \approx 0.32$, suggesting that on approaching u_c the average interface energy should scale as

$$\langle E(p, u, L) \rangle = A|u - u_c|^\mu L + BL^{\omega_c} . \tag{3.1}$$

Our numerical estimate is $\mu = 0.93 \pm 0.08$. By further considering that the critical interface is still linear, since its mass, $M(L)$, scales as $M(L) \propto L$, we find here remarkable coincidences with the undiluted random-bond Potts interface [6], suggesting that the observed transition should fall into the same universality class. Indeed, the linearity of the interface and the value of μ compatible with unity suggests that the randomness produces Ising critical behaviour independent of q as discussed in [6]. This conclusion is corroborated by a study of the probability $P_l(n)$ that the interface forms a loop at stage n. As in the undiluted case, this probability tends to 0 as n approaches infinity.

In the critical percolating regime, $p = p_c$, both absent and present bonds percolate and form incipient macroscopically spanning clusters which have strong effects on the interface shape. Again two phases exist, separated by a critical value, u_c, of u.

The interface is linear in the region above u_c and the energy distribution function flows under RG to a phase characterized by a mean $\langle E(L) \rangle \propto L$ and a width $\Delta E(L) \propto L^\omega$, with $\omega = 0.48 \pm 0.02$, peculiar to a directed path confined on an essentially one-dimensional structure (the percolation critical backbone). This is precisely the regime found for the linear DPRM at the directed percolation threshold in [9].

When $u < u_c$ a dense aggregate of bubbles forms, saturating all paths which connect the ends of the DHL, hence invading the whole percolation backbone of present bonds. The energy has a distribution whose mean and width scale as L^y and L^ω, respectively, with $y = \omega = 1.305 \pm 0.006$. Since the backbone has fractal dimension $\bar{d} = 1.30575 \ldots$†, we argue that the critical percolation geometry strongly controls the interface energy fluctuations. Indeed these are of the same order as those of the percolation cluster mass ($\omega \simeq \bar{d}$). The fact that the backbone mass and its fluctuations alone determine the interface energy scaling properties is confirmed by the results for $u = 0$ (all bond energies equal and negative), which yield the same $\omega \simeq \bar{d}$. Thus, backbone mass fluctuations scale like the average mass, and the randomness in bond energies does not add new features to the interface energy fluctuations.

At the transition point, $u = u_c = 0.661 \pm 0.001$, the scaling exponents of $\langle E(L) \rangle$ and $\Delta E(L)$ versus L are, respectively, $y_c = 0.50 \pm 0.02$ and $\omega_c = 0.51 \pm 0.01$. On approaching u_c the average interfacial energy fulfils a scaling of the form:

$$\langle E(p, u, L) \rangle = A|u - u_c|^\mu L + BL^{\omega_c} \tag{3.2}$$

with $\mu = 0.94 \pm 0.04$, as illustrated by the collapse of data in figure 3. This value is not incompatible with the undiluted froth model one [6]. However, collapse fits are not the only way to estimate μ. In order to investigate the nature of this transition more deeply it is also worthwhile calculating the scalings of interface mass, M, and probability of forming loops, $P_l(n)$, at level n [6]. M grows as L^{d_f}. d_f indicates the interface fractal dimension, and we find in the three different regimes:

$$u > u_c \qquad d_f = 1.000\,006 \pm 3 \times 10^{-6}$$
$$u = u_c \qquad d_f = 1.046 \pm 0.006$$
$$u < u_c \qquad d_f = 1.3054 \pm 6 \times 10^{-4} .$$

† Since $\bar{d} = \frac{\ln(n(1-p_c)/1)}{\ln(2/1)}$, where $n(1 - p_c)$ is the average number of present bonds forming the critical backbone, RG gives $\bar{d} = 1.305\,75 \ldots$.
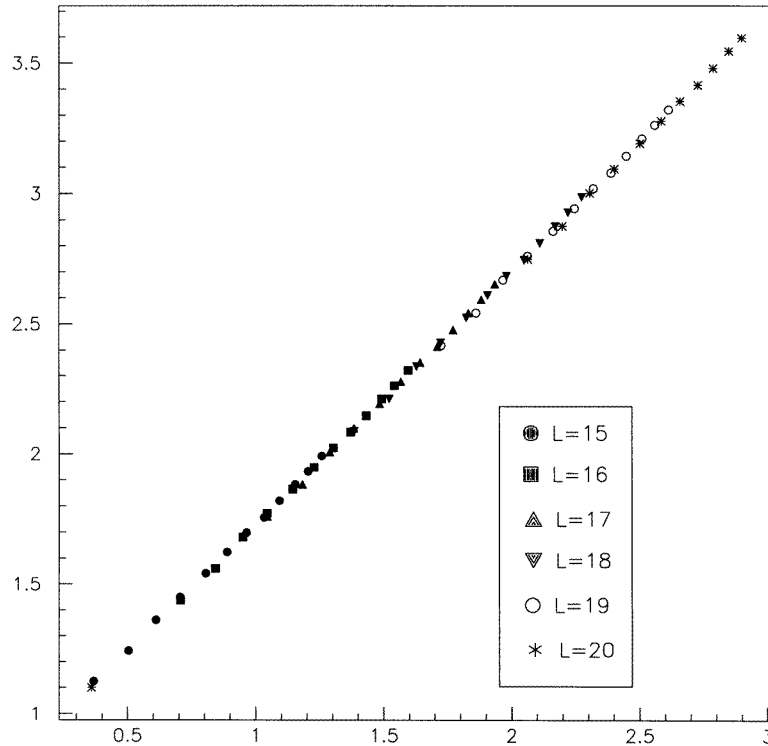
Figure 3. Data collapse of $\langle E(u, L, p = p_c) \rangle / L^{\omega_c}$ against $|u - u_c|^\mu L^{1-\omega_c}$. We measure $\mu = 0.94 \pm 0.04$ and $\omega_c = 0.51 \pm 0.01$. Scales are logarithmic on both axes.

The result for $u > u_c$ leads us to expect that $d_f = 1$ should be the exact value. Notice that, as already mentioned, $d_f \simeq \omega \simeq \bar{d}$ in the saturated phase, $u < u_c$ (see also figure 4).

The loop probability is asymptotically independent of n, $P_l \overset{n \to +\infty}{\longrightarrow} P_\infty$, and $P_\infty = 2^{d_f-1} - 1$ [6] provides an alternative way to estimate d_f. The following behaviours are found under iteration ($L = 2^n$):

$$u > u_c \qquad P_l(n) \simeq A_1 e^{-bL}$$

which implies $P_\infty = 0$, hence $d_f = 1$. Notice that A_1 and b depend, of course, on the actual u value chosen.

$$u = u_c \qquad P_l(n) \simeq P_\infty + \frac{A_2}{L^\gamma}$$

with $\gamma = 0.40 \pm 0.01$, $P_\infty = (2.3 \pm 0.6) \times 10^{-2}$, thus $d_f = 1.03 \pm 0.02$;

$$u < u_c \qquad P_l(n) \simeq P_\infty - A_3 e^{-cL}$$

with $P_\infty = 0.236 \pm 0.0009$, which implies $d_f = 1.306 \pm 0.001$. Unlike P_∞, A_3 and c show a dependence on u.

Obviously there is a linear interface for $u > u_c$, while the interface occupies the whole percolation backbone when $u < u_c$. Indeed $d_f \simeq \bar{d}$. Most interesting is the fact that the
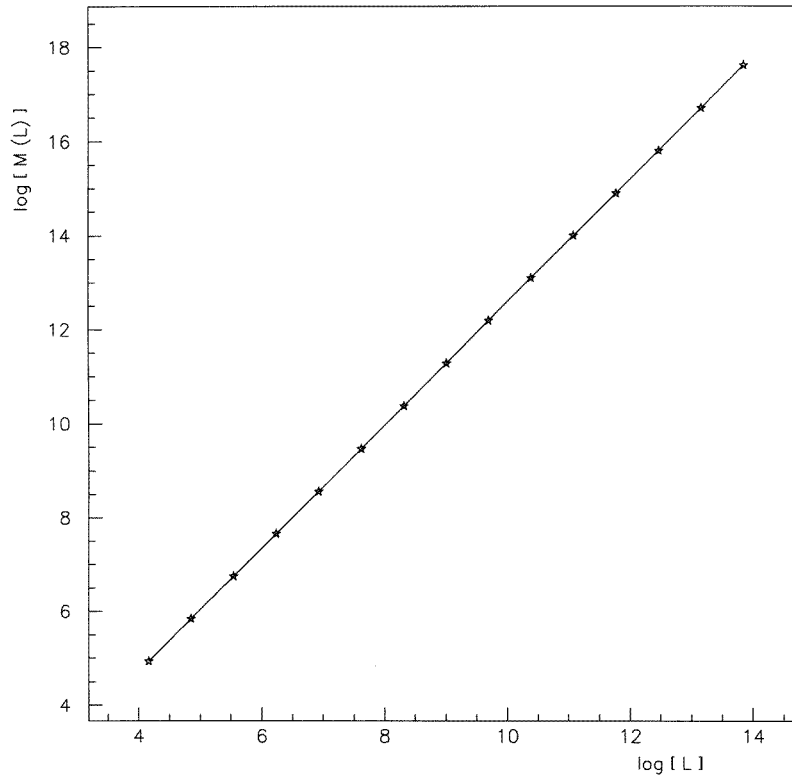
Figure 4. Plot of the interface mass, M, versus lattice longitudinal length, L, in the saturated phase, $u < u_c$, and at critical dilution $p = p_c$. M equals the directed percolation backbone mass, which thus exhibits the same scaling against L.

interface is no longer linear at the critical point $u = u_c$, because d_f is significantly different from 1, in that case, and the loop formation probability does not behave as a marginal field, as happened in the undiluted case [6]. Since P_l is a relevant scaling field, logarithmic corrections existing in the undiluted model disappear, and the critical interface is a branched object, thus indicating that we are in the presence of a different, new type of criticality.

Moreover, the RG mappings of $P_n(Z)$ at finite temperature evolve towards $T = 0$ distributions, both within stable phase regions of parameter space and on approaching the critical surface, showing that these phase transitions are also governed by zero-temperature strong-disorder fixed points, as anticipated.

For $p > p_c$ there are, of course, no paths joining the ends of the DHL, because accessible bonds do not percolate any more.

Let us now discuss in more detail the percolative landscape across which the interface develops when $p = p_c$. The probability of a present bond is $\varrho = 1 - p$, which transforms under RG as

$$\varrho_{n+1} = \varrho_n^2(2 - \varrho_n)^2. \tag{3.3}$$

The fixed point $\varrho_c = 1 - p_c$ sets the percolation threshold. So at $p = p_c$ a critical cluster of present bonds also exists, as already noticed. Considering the anisotropy of DHL, which naturally introduces a preferred direction, the longitudinal one, for the percolation process

(RG recursions (2.4) and (3.3) are explicit consequences of such a directedness), and taking into account the DHL fractal dimension, equal to 2, the critical percolating cluster on it is eligible to represent a sufficiently faithful approximation of the actual structure of the 2D directed percolation cluster. The better such a circumstance can be confirmed, of course, the more value should be added to our investigation of interfacial properties on DHL. Of course, there are limits to such an attitude: it should also not be forgotten, for example, that isotropic and directed percolation are not distinguishable on the DHL.

Since the Potts interface in its saturated phase occupies the whole directed percolation backbone, we are able to use such an interface as a probe of backbone structure by the RG analysis. The directed backbone fractal dimension, $\bar{d}$, has already been determined both analytically (see footnote on page 2444) and numerically by evaluating the saturated interface mass scaling, shown in figure 4. We found a value surprisingly close to the one known for two-dimensional directed percolation ($\bar{d} \approx 1.31$ [14]).

The correlation exponents of percolation, $\nu_\parallel$ and $\nu_\perp$, are equally accessible on DHL. From (3.3) we calculate the rescaling of $\Delta\varrho = \varrho - \varrho_c$:

$$\Delta\varrho' = \lambda_c \Delta\varrho = b^{y_\varrho} \Delta\varrho \qquad \lambda_c = 6 - 2\sqrt{5}. \tag{3.4}$$

The two correlation lengths, parallel and normal to the preferred direction, diverge near ϱ_c as

$$\xi_\parallel \sim |\varrho - \varrho_c|^{-\nu_\parallel} \qquad \xi_\perp \sim |\varrho - \varrho_c|^{-\nu_\perp}. \tag{3.5}$$

Manifestly, on this hierarchical lattice $\xi_\parallel$ concerns the correlation in the longitudinal direction connecting the two ends of the lattice. Thus, if $\xi_\parallel' = b^{-1}\xi_\parallel$, with $b = 2$ (length rescaling factor), it follows:

$$\frac{1}{\nu_\parallel} = \frac{\ln \lambda_c}{\ln b} = 0.611\,516\,17\ldots \quad \text{and} \quad \nu_\parallel = 1.635\,279\,7\ldots$$

(the two-dimensional exponent is slightly larger, $\nu_\parallel = 1.735$ [15]).

To derive $\nu_\perp$ we define the distance along the direction orthogonal to the lattice as equal to the number of branching levels separating two given sites. With such a convention the average transverse size of the percolation backbone, $L_\perp$, then transforms according to

$$L_\perp^{(n+1)} = \frac{2\varrho_n^2(1-\varrho_n)^2 L_\perp^{(n)} + 2\varrho_n^3(1-\varrho_n)3L_\perp^{(n)} + \varrho_n^4 2L_\perp^{(n)}}{\varrho_{n+1}}. \tag{3.6}$$

Applying finite-size scaling to (3.5) we get $\xi_\perp \sim L^{\nu_\perp/\nu_\parallel}$, at $\varrho = \varrho_c$. From equation (3.6) it follows $L_\perp^{(n+1)}|_{\varrho_c} = 4\varrho_c^2 L_\perp^{(n)}|_{\varrho_c}$, then $\nu_\perp/\nu_\parallel = \ln 4\varrho_c^2/\ln 2 = 1/\nu_\parallel$, and $\nu_\perp = 1$. These results should be compared with the two-dimensional exponents: $\nu_\perp/\nu_\parallel = 0.633$; $\nu_\perp = 1.097$ [15].

Finally the simplicity of our RG also allows for iterating the number of steps on which the saturated interface is linear, and which therefore connect different blobs of the directed backbone. From a percolation point of view these are just the 'red bonds' forming one-dimensional links of the directed backbone. Like in isotropic percolation [16] one can show that the directed backbone red bond mass, M_R, at ϱ_c has a scaling form $M_R \propto L^{D_R}$, with $D_R = 1/\nu_\parallel$, when the system has linear size L. We measured a numerical value $D_R = 0.6111 \pm 0.0006$, which agrees fairly well with the above theoretic prediction of $1/\nu_\parallel$ [17]. Figure 5 shows a plot of M_R versus L.

The percolation critical exponents calculated above for our simplified model of directed percolation are remarkably consistent and close to the truly two-dimensional ones, that we hope the same could apply to the scaling properties of the Potts interface we have determined here.
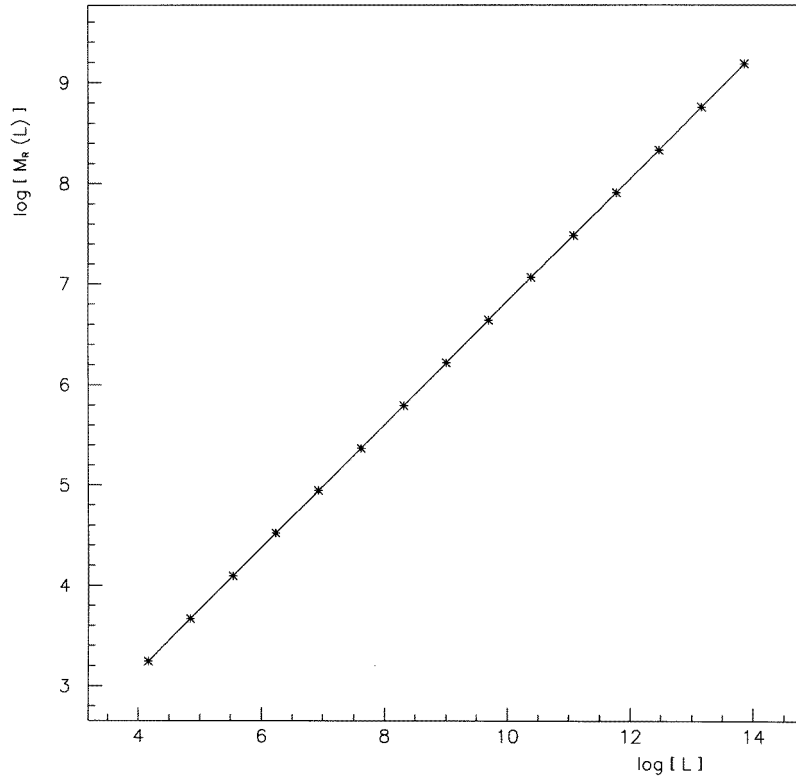
Figure 5. Plot of the 'red bond' mass, M_R, in the critical percolation backbone, versus the lattice longitudinal length L.

4. Conclusions

In this work we addressed the critical behaviour of a Potts ferromagnet in the presence of both exchange disorder and dilution. The scaling properties of the interfacial free energy were considered and dilution acted in the form of percolative regions not accessible to the interface, due to extremely strong ferromagnetic couplings. By restricting interfacial configurations to those which can be hosted by a DHL, an RG analysis of the line tension can be carried on through a very accurate numerical procedure. For all dilution regimes considered the interface scaling turns out to be controlled by $T = 0$ fixed-point distributions. This makes the analysis relatively easier, since $T = 0$ recursions for the interfacial energy can be studied more effectively than their $T > 0$ counterparts.

When accessible regions are above their percolation threshold ($p < p_c$), dilution does not seem to modify interfacial scaling with respect to the undiluted case. The three regimes occurring in this case are those on the basis of which the Ising-like nature of criticality in disordered Potts ferromagnet in 2D could be argued [6]. New scalings occur when dilution is at threshold and the interface has to develop within the incipient infinite cluster backbone. When, at $T = 0$, u is such that it guarantees a sufficient dominance of positive energy bonds the interface remains linear at large scales and appears to behave in the same way as a strictly linear DPRM would in such an environment [9]. As in the corresponding regime above percolation threshold, ramification is fully inhibited if $u > u_c$. However,

the exponent $\omega \simeq 0.48$ falls in a new universality determined by the backbone geometry limiting the linear interface [9].

A completely new behaviour is realized right at $u = u_c$ when dilution is at threshold. In such conditions we get clear evidence that the interface has non-zero probability of branching and thus behaves as a fractal. Its dimension slightly, but definitely exceeds 1. Of course, by construction, on DHL a strictly linear interface cannot behave as a fractal. However, a fractal dimension equal to 1 would persist in the linear regime when replacing the DHL with more complicated lattices allowing, in principle, for a fractal linear DPRM geometry.

The above interface scaling regime at $u = u_c$ is the first critical one with non-trivial fractal geometry met so far in this kind of study, and indicates the highly non-trivial effect that backbone geometry can exert on the interface and on the critical behaviour of the system. So far, borderline regimes between the linear and maximally branched ones in such froth models were never seen to allow for non-zero, intermediate looping probability. Thus the critical behaviour of the interface could not be seen to be different from the linear, Ising-like one at low-T [6]. The values of μ and d_f estimated for this regime lead to $\nu = 1/\mu \approx 1.06$ or $\nu = d_f \approx 1.046$ for a critical Potts model subject to threshold dilution and disorder. This can be concluded within the same limits of the arguments developed in [6]. Like in that undiluted case, the value of ν, being associated to a $T = 0$ fixed distribution, is universal with respect to q.

In the last regime, with critical dilution and $u < u_c$, we could realize how the interface can become a sort of probe for the structural properties of the backbone. The interface mass M scales as the backbone mass with longitudinal distance. Quite remarkably, the energy fluctuations are fully determined by backbone mass fluctuations here. These backbone mass fluctuations in turn scale with the same exponent as the total average mass.

As an instructive exercise adding credibility to our DHL model as a good qualitative picture of the situation on a 2D Euclidean lattice, we produced here results for $\nu_\parallel$ and $\nu_\perp$ and the 'red bond' dimension of directed percolation. No similar determination of $\nu_\perp$ was produced before, to our knowledge. These results altogether appear remarkably consistent with the numerical estimates in 2D [15].

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